GAUSS ACADEMY of Mathematical Education
MATH UNITES US

Gauss School and Gauss Math Circle

2021 Gauss Math Tournament

Division III (Sprint Round 50 minutes, 40 Questions)

1. Gina is inviting 4 boys and 6 girls to her party. If there are 7 girls and 6 boys in her class, how many ways are there to choose who to invite? (105)

2. Two green balls and five red balls are placed in an urn. Two random balls are chosen from the urn, without replacement. What is the probability that both are red? **(CQ)**
Answer: **10/21**

3. In a certain month, there are three Sundays with even-numbered dates. What day of the week is the 18^{th} of this month? **(CQ)**
Answer: **Tuesday**

4. Gauss High School has 168 freshmen, 134 sophomores, 124 juniors, and 54 seniors. What percentage of the students at Gauss High School are freshmen? **(CQ)**
Answer: **35%**

5. Ayaka, Benjamin, and Cheung are playing a coin game. Each person repeatedly tosses a fair coin, and stops when he/she tosses heads for the first time. What is the probability that Ayaka, Benjamin, and Cheung all tossed the same number of tails? **(CQ)**
Answer: ***1/7***

6. The area of the parallelogram ABCD is 56 square centimeters. E and F are the midpoints of AB and AD, respectively. Find the area of triangle CEF. **(CQ)**
Answer: ***21 square centimeters***

7. How many degrees does the hour hand of a regular clock travel between 2:30 PM and 2:50 PM on the same day? **(CQ)**
Answer: ***10°***

8. Let ABC be a triangle with AB=3, BC=4, CA=5. Find the distance between the midpoints of the A-median and the C-median. **(DX)**
Answer: ***5/4***

9. A triangle ABC is intersected by a line l at point F on AB, point E on AC, and point D on the extension of BC. If triangle ABD has a right angle at A, BC=CD=13, AD=24, and DE bisects angle ADB, what is the length of AF? **(DZ)**
Answer: ***24/5***

10. Steve draws an equilateral triangle T. He then draws square A with the same perimeter as T, and square B with the same side length as T. What is the ratio of the area of A to the area of B? **(DX)**
Answer: ***9/16***

11. In the sequence, $A_1 = 2$, $A_2 = 15$, and for all positive integers, $A_{n+2} = A_{n+1} - A_n$. What is A_{2021}? **(CQ)**
Answer: ***-15***

12. Find $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+...+\frac{1}{97\cdot 99}$. **(CQ)**
Answer : ***49/99***

13. There are three positive integers x, y, z, so that x+y=2021, z-x=2020, and x<y. What is the maximum possible sum of x, y, z? **(CQ)**
Answer: ***5051***

14. Suppose that a sequence $x_n$of integers satisfies $x_0 = 1$and $x_n = x_{n-1} - n$if $x_{n-1} \geq 0$and $x_n = -x_{n-1}$if $x_{n-1} < 0$. Find x_{2021}. **(AL)**
Answer: ***1010***

15. Given that x^3-x=c has two real roots and c is a positive number, find the positive root. **(DZ)**
Answer: ***2sqrt(3)/3***

16. Hunter writes down two real numbers such that their sum and product are equal. There exists a single real number x that Hunter could not have possibly written. What is x? **(DX)**
Answer: ***1***

17. What is the smallest perfect square that is two more than an odd prime? **(CQ)**
Answer: **9**

18. Mr. Pickles likes to play with numbers and pickles, but mostly numbers. He wants to find the largest number less than 10,000 that is 2 mod 4, 5 mod 7, 9 mod 11, and 13 mod 15. What is the sum of the digits of this number? **(DZ)**
Answer: ***22***

19. How many divisors of 240 are also divisors of 420? **(DX)**
Answer: ***12***

20. Six circles are drawn in the plane. The number of regions created by the circles is S. Find the maximum possible value of S. **(CQ)**
Answer: ***32***

21. In each round of the game Math to Win! two players compete, and one will win, the other will lose. The total score of each player is the number of games they win. In a game of Math to Win! every pair of players among A, B, C, D compete in one round, for a total of 6 rounds. Given that A wins his game against D, and after all rounds are completed A, B, C have the same score, what is the score of D? **(CQ)**
Answer: ***0***

22. Kyle has a 15-centimeter-long piece of wire. She cuts the wire into three pieces, each of which has a positive integer length. How many non-congruent triangles can be formed by the three pieces of wire? **(CQ)**
Answer: ***7***

23. A group of 11 students want to board two buses: Bus A which can hold up to 8 people and Bus B which can hold up to 3 people. Four of the students are Ada, Ben, Cat, and Dan. Given that Ada and Cat want to be on the same bus and Dan and Ben cannot

both board bus A, how many ways are there for all 11 students to be assigned to a bus? **(CQ)**
Answer: ***51***

24. Alan has a square paper ABCD with side length 4. The paper is folded along BD until the dihedral angle between planes ABD and CBD is 60°. In the resulting figure, what is the distance between A and the midpoint of BC? **(CQ)**
Answer: ***2sqrt2***

25. Let a be a positive integer. There exists exactly one positive integer b such that there exists a nondegenerate triangle with side lengths 5, a, and b. Find all possible values of a. **(DX)**
Answer: ***1***

26. Let $x_1, x_2..., x_n$be positive integers so that $x_1 + x_2+...+x_n = 19$. Maximize the value of $x_1x_2...x_n$. **(AL)**
Answer: ***972***

27. Given that the sequence $\{A_n\}$satisfies $A_1 = 2$ and $A_{n+1} = \frac{1+A_n}{1-A_n}$ for all natural numbers$n$, find $A_1A_2A_3...A_{2021}$. **(CQ)**
Answer: ***3***

28. Find the sum of all solutions to 2-3x=||x|-1|. **(CQ)**
Answer: ***1/2***

29. Given that x and y are nonnegative real numbers satisfying $x + y = 10$, minimize the value of $3x^2 + xy + 2y^2$. **(AL)**
Answer: ***575/4***

30. Suppose that $f(x)$is a function $Q^+ \to Z$(from the positive rational numbers to the integers) which satisfies $f(1) = 0$and $f(xp) = f(x) + p$for any positive rational number $x$and prime p. Find the value of $f(\frac{2020}{2021})$. **(AL)**
Answer: ***20***

31. Find the sum of all integers n>8 such that n^2 is divisible by n-8. **(DX)**
Answer: ***183***

32. Find the largest positive integer $z$so that $z$cannot be expressed as $7a + 8b + 9c$for some nonnegative integers a, b, c. **(AL)**
Answer: *20*

33. Suppose that $x \equiv a \ (mod\ 7), x \equiv b \ (mod\ 11)$for some positive integer x. Then $x \equiv x_1 a + x_2 b \ (mod\ 77)$for all x, where $x_1, x_2$are integers from $0$to 76inclusive. Find $x_1 + x_2$. **(AL)**
Answer: ***78***

34. Find the last two digits of 2021^{2021}. **(AL)**
Answer: ***21***

35. How many nonempty subsets of {1,2,4,8,16,32} have the property that the product of the elements is divisible by the sum of the elements? **(DX)**
Answer: ***6***

36. The sum of the divisors of N is equal to the sum of the divisors of 16, where N is a positive integer not equal to 16. Find N. **(DX)**
Answer: ***25***

37. Andrew participated in the USA Multiplication Olympiad (USAMO). There are 6 problems. For each problem, Andrew gets an integer score from 0 to 7, inclusive. However, his final score is determined by multiplying the scores of the individual problems, not adding them together. How many possible scores could Andrew receive? **(AL)**
Answer: *463*

38. Let a be a positive real number. The lines y=ax, y=2ax, and x+y=1 bound a triangle. Find the largest possible area of this triangle. **(DX)**
Answer: ***(3-2sqrt2)/2***

39. In the cube ABCD-A1B1C1D1 (with edges AA1, BB1, CC1, DD1) with side length 6, E is the midpoint of AB. What is the distance between lines DE and B1C? **(CQ)**
Answer: *2sqrt(6)*

40. A right triangle has three integer side lengths. Moreover, the length of the altitude to the hypotenuse is also an integer. What is the smallest possible value of the length of this altitude? **(DX)**
Answer: ***12***

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Division III (Target Round 20 minutes, 8 Questions)

1. A seven-digit number is ultra-friendly if it satisfies both of the following two conditions:
(1) it contains digits 1,2,3,4,5,6,7 exactly once each;
(2) no three consecutive digits are all odd.
How many ultra-friendly numbers are there? **(CQ)**
Answer: ***2736***

2. You are playing a game with Cheating Chika. Each of you flips five coins, and you win if you flip strictly more heads than Chika. However, Chika decides to cheat, and one of her coins has both faces heads (her other four coins, and all five of your coins, are fair). What is the probability that you win? **(DX)**
Answer: ***65/256***

3. A triangle ABC has side lengths AB=7, BC=8, CA=9. Let X be the intersection of the line perpendicular to BC from point A and the median from B to AC. Let D, E, F be the centroids of triangles XBC, XAC, XAB, respectively. Find the square of the area of triangle DEF. **(CQ)**
Answer: ***80/9***

4. Triangle ABC has $\angle A = 60$degrees and $BC = 11$. Points D and E are on sides AB and AC, respectively, such that BD=CE and $DE = 7$. Compute the area of quadrilateral $BDEC$. **(AL)**
Answer: *18sqrt(3)*

5. A triangle with side lengths 3, 4, 5 and a 1x100 rectangle overlap. Find the largest possible area of their intersection. **(DX)**
Answer: ***95/24***

6. A sequence a_n satisfies $a_1 = 53, a_2 = 7, (53 \cdot 7)({a_i}^2 - a_{i-1}a_{i+1}) = a_i a_{i-1}$ for all $i \geq 2$. Compute the first value of k so that $a_k = 0$. **(AL)**
Answer: *51*

7. A random integer n is selected uniformly and at random from 1 to 729, inclusive. Let k be the largest integer for which n/3^k is an integer. Find the expected value of k. **(DX)**
Answer: ***364/729***

8. Compute the smallest positive integer k so that the smallest number with 5^k factors has a factor of the form a^{10}, where a is an integer greater than 1. **(AL)**
Answer: ***12***