2023 Gauss Math Tournament Sprint Round (Div. 3)

June 10, 2023

1. Define binary operation $\otimes$ between ordered pairs by $(a,b)\otimes(c,d)=(a+d,ad-bc)$ for all real numbers a, b, c, d. What is the sum of the coordinates of $((6,5)\otimes(4,3))\otimes(2,1)$?

2. The mean and mode of the set 2, 3, 7, 5, x, 6, x are the same number. Find x.

3. Sana has an unknown number of cookies in her cookie jar. She takes out half of the cookies from the jar and gives it to her little brother, Haya. Haya adds these cookies to his stash, which previously has 100 cookies in it. Afterwards, he splits his cookie stash evenly among his seven best friends. If each of his friends received 16 cookies, how many cookies did Sana have at the start?

4. After buying some more cookies, Sana refills her cookie jar. If she had bought four more cookies, she could have split all her cookies evenly into seven smaller jars. What is the largest possible number of cookies Sana could have if she has less than 150 cookies?

5. In his first five ever golf games, Charlie shot scores of 85, 80, 77, 88, and 75. If he wants to lower his average score by 2, what does he have to get on his next game?

6. Seven identical markers are to be split up into three classes: English, Math, and Science. If each class has at least one marker, how many ways are there to split the markers into the classes?

7. A cube with side length 7 is painted on all faces before being cut into 343 unit cubes. How many of these unit cubes have at least one face painted?

8. Points D, E, F lie on sides AB, BC, CA, respectively, of equilateral triangle ABC. If $\frac{AD}{DB}=\frac{BE}{EC}=\frac{CF}{FA}=\frac{1}{6}$, find $\frac{[DEF]}{[ABC]}$ (for a polygon $\mathcal{P}$, denote $[\mathcal{P}]$ to be its area).

9. The area of the circumcircle of the triangle with vertices at $(0,0)$, $(1,2)$, and $(-2,1)$ can be expressed as $k\pi$. Find k in simplest form.

10. How many positive divisors does 9072 have?

11. Let x denote the answer to this question. What is x^3+3x^2+4x+1?

12. Triangle $\triangle ABC$ has orthocenter H. Let D be the foot of the altitude from B to AC. If $\frac{[AHB]}{[BHC]}=\frac{1}{7}$, find $\frac{CD}{DA}$.

13. Let f be a function for which $f(x^3+4)=x^2+4x+4$. What is $f(733)$?

14. A positive integer N satisfies the following:

$$\begin{cases} N\equiv 3 \pmod{5} \\ N\equiv 5 \pmod{7} \\ N\equiv 15 \pmod{17} \end{cases}.$$

What is the smallest possible value of N?

15. Three points are randomly chosen from the perimeter of a regular pentagon. What is the probability that they form a nondegenerate triangle?

16. The area of the region enclosed by $(|x|-3)^2+(|y|-3)^2 \leq 12$ can be represented as $a\pi + b\sqrt{c}$ in simplest radical form. Find $a+b+c$.

17. A circle whose center has coordinates of $(0, m)$ with $m < 5$ is tangent to the lines $y = x$, $y = -x$, and $y = 5$. If m is of the form $a - b\sqrt{c}$, what is abc?

18. What are the last two digits of $2023 + 7^{2023}$?

19. Square $ADEF$ is inscribed in right triangle $\triangle ABC$ such that D lies on AB, E lies on BC, and F lies on CA. If the side lengths of $\triangle ABC$ are 6, 8, and 10, what is the side length of $ADEF$?

20. Find the number of positive integers $n \geq 2$ such that the remainder when 2023 is divided by n is equal to the remainder when n is divided by 3.

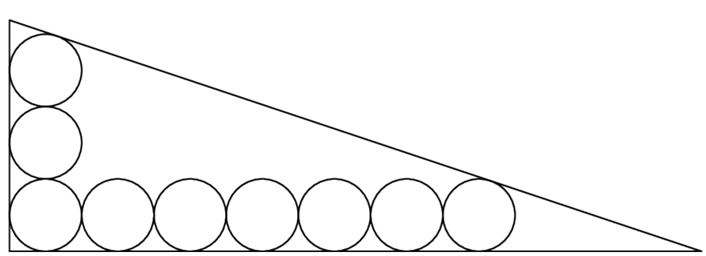
21. There exists a positive real value n such that the equation $nx^2 = x + n$ has two real solutions for x that differ by five. What is n^2?

22. Bob is trying to pass a true/false test with 35 questions. To pass, Bob needs to earn a score of 70% or higher, where each question is worth one point. Bob answers N questions correctly, guesses on the remaining $35 - N$ questions, and realizes that his probability of passing the test is greater than 50%. What is the smallest possible value of N?

23. Triangle $\triangle ABC$ has $AB = 30$, $AC = 40$, and $BC = 50$. Point D outside of $\triangle ABC$ is constructed such that $\angle BDC = 45°$ and $CD = 1$. The distance between A and the circumcenter of BCD can be written as $a\sqrt{b}$ in simplest radical form. Find $a + b$.

24. A land surveyor stands in the lower left cell of an 8×16 grid. Every minute, he walks one unit in an arbitrary direction. He visits each cell exactly once before stopping, and he never leaves the grid. In minutes, the minimum and maximum time it could take him to survey every cell exactly once are m and M. Find $m + M$.

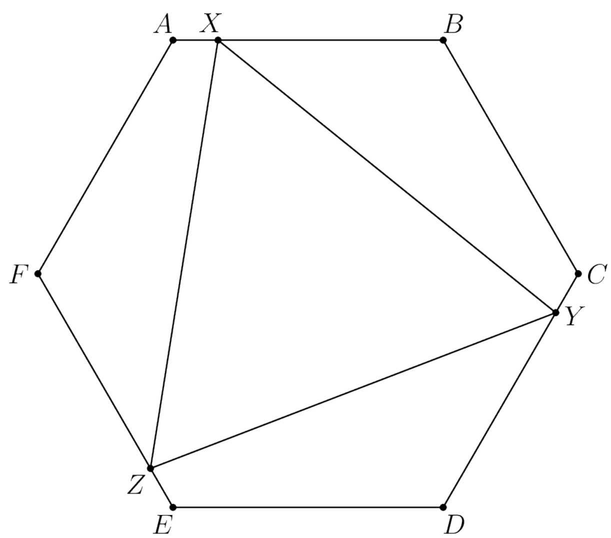
25. In the diagram below, all nine circles are congruent and the length of the longest side of the right triangle is 30. Compute the area of the triangle.



26. How many subsets of $\{1, 2, 3, \ldots, 11\}$ don't have any consecutive integers in them?

27. The real root of the polynomial $12x^3 + 12x^2 + 48x + 64$ can be written in the form $\frac{a}{\sqrt[3]{b}+c}$, where a, b, c are integers. Find $a + b + c$.

28. In the figure below, $ABCDEF$ is a regular hexagon with side length six. Given that $AX = CY = EZ = 1$, and the area of XYZ is in the form of $\frac{a\sqrt{b}}{c}$ such that a and c are relatively prime, what is $a + b + c$?



29. A cube with side length 9 is painted on all faces and split into 729 unit cubes. The unit cubes are then randomly rearranged into a new cube with side length 9, and once again painted on all faces. Compute the expected number of unit cubes that have paint on all six faces.

30. What are the last three digits of $2023 + 7^{2023}$?

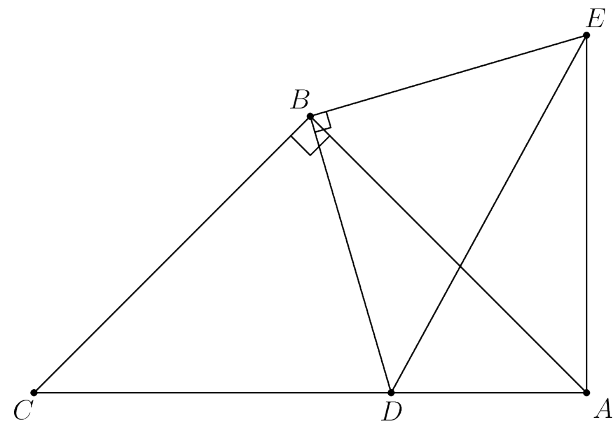
31. Let z_1, z_2, z_3, z_4 be all complex roots of $z^4 - 6z^2 + 19z = 0$. Find $z_1^3 + z_2^3 + z_3^3 + z_4^3$.

32. P and Q lie on edges AA_1 and CC_1 of unit cube $ABCD - A_1B_1C_1D_1$. If $AP = C_1Q = \frac{1}{3}$, find the area of parallelogram $PBQD_1$. The answer can be expressed as $\sqrt{a}/b$, where a and b are integers and a is square-free. Find $a + b$.

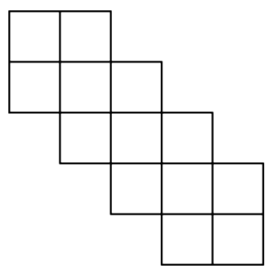
33. Very bored, Julie plays with her custom calculator. Her custom calculator has two keys: $\times 2$ and $+1$. Her screen shows the number 1. She wants to figure out the smallest and largest number of keystrokes it could take her to get the number 2023. If these numbers are m and M, respectively, find M/m in simplest form.

34. In triangle ABC, $AB < BC < AC$. Two circles, ω_1 and ω_2, are constructed with centers at the circumcenter of ABC such that ω_1 is tangent to AB and ω_2 is tangent to BC. ω_1 intersects AC at points D and E, such that D lies between A and E, and ω_2 intersects AC at points F and G, such that F lies between A and G. If $AD = 99$, $DF = 100$, and $FG = 101$, compute $BC^2 - AB^2$.

35. In the figure shown, $\triangle ABC$ and $\triangle DBE$ are both right isosceles triangles with a right angle at B, and D lies on AC. If $AB = 2AD = 3$, what is $AE - CD$?



36. Let A_i denote the number of ways to place i knights in the board below such that no two knights attack each other. Compute $A_5 + A_7 + A_9$.



37. Let $m = \overline{ABCDE}_3$ and $n = \overline{FGHIJ}_4$ be two positive integers satisfying $\overline{ABCDE}_{10} = 2\left(\overline{FGHIJ}_{10}\right)$ for not necessarily distinct digits A through J, and $A \neq 0$. Find the maximum possible value of $n - m$.

38. The nation of Arcadia is a straight line bordered by two cliffs on each endpoint. Forty-one towns labeled $-20, -19, \ldots, -1, 0, 1, \ldots, 19, 20$ lie in Arcadia from East to West, and there are forty roads connecting each pair of consecutive towns and two roads connecting the outermost towns to their closest cliff. Riell is at town -10 on Day 1. Every day thereafter, she travels one of the two roads connecting the town to another town or a cliff. If she falls off a cliff, she leaves Arcadia, never to return. What is the probability that Riell reaches town 0 before leaving Arcadia?

39. On an 8 by 8 grid, Chris selects one unit square at random and colors it blue. The expected number of squares of any size with edges along the grid lines containing the blue square can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

40. Positive real numbers x, y satisfy $[xy] - [x][y] = 6$, where $[x]$ denotes the greatest integer function. Find the sum of all possible values of the quantity $[2xy] - [2x][y]$.