Gauss School and Gauss Math Circle

2019 Gauss Math Tournament

<u>Grade 7-8 (Sprint Round 50 minutes, 40 Questions)</u>

1. Suppose x is the answer to this question 2^x=1. Compute $-x$.

2. The amount of ways to arrange the letters in BANANA is n, and the amount for BANANAS is m. Compute m/n.

3. Compute the area of a triangle with side lengths $13, 14$ and 15.

4. John flips 3 fair coins. What is the probability that at least 1 of them is a head?

5. How many numbers under 100 are divisible by 3 or 4?

6. A sphere, when measured around its widest part, has circumference of 10 units. Compute its surface area.

7. An ant is at point $(0, 0)$, and wishes to get to $(5, 5)$. If it can only go up one unit or right one unit every turn, how many ways can it get there?

8. How many ways are there to color exactly 4 unit cubes from a $2 \times 2 \times 2$ cube, if rotations are indistinct?

9. Farmer John plants cauliflower and raises cattle. Cauliflower have 1 head and no legs, whereas cows have 1 head and 4 legs. On the farm, there are a total of 2019 heads and 2016 legs. How many cauliflower does he have?

10.The sum of the lengths of the legs of a right triangle is 2019, and the hypotenuse has length 1619. Compute the area of the triangle.

11.Bessie the Cow is sailing from Valery-sur-Somme, France to Pevensey, England. It is 100 miles between the two towns, and a strong wind blows south, parallel to her journey. If it takes her 10 hours to arrive in Pevensey, and 4 to return to Valery-sur-Somme, compute the wind speed in mph.

12. Let $f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$. Compute $\underbrace{f(f(f(\dots f(2)\dots)))}_{2019\ times}$

13. There exists a 2 digit integer n such that $n^3 = 110592$. What is n?

14. One root of the polynomial $x^2 - 2016x + 871664$ is $x = 628$. Compute the other root.

15. Noobmaster will play Fortnite with probability 100% if Thanos does not snap his fingers, and 50% if he does. If the probability that Thanos snaps his fingers is p, what is the maximum value of p such that Noobmaster will play Fortnite with probability at least 99%?

16. To make an iPhone password, one must choose a 4 digit string of numbers 0-9, which contains at least one double number (i.e. a number which occurs twice consecutively). How many valid iPhone passwords exist?

17. A polynomial with integer coefficients satisfies $P(1) = P(2) = P(3) = 7$. Find the minimum positive value of $P(10)$.

18. p, q, r are primes such that $p + pq + pqr = 2019$. Compute $p + q + r$.

19. Evaluate the infinite sum below:

$$\frac{1}{1\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 9} + \frac{1}{7\cdot 11} + \cdots$$

20. Let T be the answer to the previous question. Evan uniformly choose 2 real numbers a and b from the intervals $[0, 2T)$ and $[0, 3T)$ respectively. What is the probability that $a + b < T$?

21. A $3 \times 3 \times 3$ cube is divided into 27 $1 \times 1 \times 1$ cubes, and each cube is colored either red, green, or blue, with equal probability. What is the expected amount of pairs of adjacent cubes with the same color?

22. Find the smallest k such that 1001 divides $2^k - 1$.

23. Triangle ABC has $BC = 5$, $AC = 7$, and $AB = 8$. If the incircle touches sides BC, AC, and AB at D, E, and F respectively, compute the area of triangle DEF.

24. A cubic P satisfies $P(1) = 1$, $P(2) = 4$, $P(3) = 9$, and $P(5) = 121$. Compute $P(4)$.

25. If $a\&b = \frac{ab}{a+b}$, what is $1\&(2\&(4\&(8\&(\ldots(1024\&2048)\ldots))))$?

26. How many ways are there to cover a 4×4 square with only 2×2 and 1×1 squares, if tiles cannot be cut, exceed the boundary of the big square, or overlap?

27. 100 chicks sit in a circle. At a given time, all of them peck 1 of their 2 neighbors with equal probability. At a later time, with the chicks still in the circle, only the unpecked chicks will peck 1 of their 2 neighbors, with equal probability. Considering only the second round of pecking, how many chickens

are unpecked? (i.e. count chickens that were pecked first round, but not second round.)

28. Starting from the origin, a bug will travel 1 unit in one of the 4 cardinal directions every second, each with equal probability. After 22 seconds, what is the expected value of the square of its distance from the point $(29, 34)$?

29. William has a unit square $ABCD$. He first draws a line parallel to AB, which is chosen uniformly between AB and CD, and then a line parallel to BC, chosen uniformly between BC and AD. If the intersection of these two lines is P, what is the probability that the sum of the squares of the distances from P to A, B, C, D is at least $10/3$?

30. If $x^2 + x + 1$ divides $x^{10} + ax^8 + 5x^5 + bx^2 + cx$, then compute $a + b$.

31. Find the largest integer x such that $x^2 + 57x + 2870$ is a perfect square.

32. What is the smallest period in the decimal expansion of $\frac{1}{243}$?

33. Leo has a unit cube $ABCDEFGH$, with faces $ABCD, EFGH, ABFE, CDHG$. He chooses, at random, a square of side length 0.25 completely on side $ABCD$, with sides parallel to the sides of the cube, and bores a hole through this square, which goes all the way to the corresponding location on $EFGH$. He does this for faces $ABFE$ and $CDHG$ as well. What is the probability that the two tunnels intersect?

34. Let $f(x) = x^3 + ax^2 + 28x + c$. Given that the sum of the roots equals the sum of the squares of the roots and the product of the roots equals the sum of the cubes of the roots, find the largest possible value of c.

35. Find the only positive root of $x^4 + 4x^3 + 5x^2 = 3$.

36. William and Leo are playing a game with a pile of n tokens. William goes first. On a player's turn, if there are m tokens, he may take k tokens away from the pile as long as k is a prime divisor of m. If m is prime on the player's turn, that player loses. For how many integers $1 < n < 2019$ can William win?

37. In a sequence, $a_1 = a_2 = 2 - \sqrt{3}$, and $a_{i+1} = a_i + a_{i-1} + a_{i+1}a_ia_{i-1}$. Compute a_{10}.

38. Eric is taking a flight from Juneau, Alaska to Orlando, Florida. Approximate the coordinates of Juneau as $(60^oN, 125^oW)$ and Orlando as $(30^oN, 80^oW)$. Approximating the Earth as a sphere of radius r meters, if the shortest possible flight distance is d, evaluate $cos(d/r)$, where the argument is in radians.

39. The polynomial $x^4 - bx^2 - 1550x + d$ has 4 real roots. Find the sum

of their cubes.

40. Triangle ABC has $AB = 13$, $AC = 15$, and $BC = 14$. If $D \neq A$ is on segment AC such that $BD = 13$, find the ratio of the inradius of ABD to the inradius of BDC.

–End of Sprint Round–

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Grade 7-8 (Target Round 20 minutes, 8 Questions)

1. How many zeroes does 720! have in base 6?

2. 10 people play a game. In a round, everyone flips a fair two-sided coin. A winner is determined if they are the sole person to flip heads or tails, with everyone else obtaining the opposite result. Compute the expected number of rounds necessary to determine a winner.

3. The fibonacci sequence is defined by $F_1=1$, $F_2=1$, and $F_n=F_{n-1}+F_{n-2}$ for all $n>2$. Given that $F_{18}=1597$ and $F_{22}=10946$, evaluate $3 * F_{20}$.

4. Let T be the answer to the last question. Eddie uniformly and randomly chooses 2 real numbers a and b, with a in the interval [0, 2T) and b in the interval (0, 3T]. Find the probability that $a+b < T$.

5. Barney is flipping a coin. He will only stop when he flips three consecutive tails in a row. What is the expected number of flips before Barney stops?

6. Triangle ABC has sidelengths AB = 4, BC = 5, CA = 6. Let D be placed on BC such that BD = 2. Find the ratio of the inradii of triangle ABD to triangle ACD.

7. In triangle ABC, angle A is 120 degrees. If AC + AB = 15 and AC + BC = 20, find BC.

8. Let P(x) = x^3 - 10x^2 - x + 2019 and let the roots of P(x) be a, b, c. The cubic polynomial Q(x) has roots bc-a^2, ca-b^2, ab-c^2 and is monic. Find Q(-1).

End of Target Round

Gauss 7-8 2019 **Sprint** Solutions

1 Solutions

1. 0
2. 7
3. 84
4. 7/8
5. 49
6. $100/\pi$
7. 252
8. 7
9. 1515
10. 363800
11. 7.5 or $\frac{15}{2}$
12. 2
13. 48
14. 1388
15. 1/50 or 2%
16. 2710
17. 7

18. 229

19. 1/3

20. 1/12

21. 18

22. 60

23. $30\sqrt{3}/7$

24. 40

25. 2048/4095

26. 41

27. 75

28. 2019

29. $(12 - 4\sqrt{3} - \pi)/12$

30. -5

31. 2029

32. 27

33. 5/9

34. 245/2

35. $(\sqrt{5} - 1)/2$

36. 1000

37. $-2 - \sqrt{3}$

38. $(\sqrt{6} + 2\sqrt{3})/8$

39. 4650

40. 264/49

Target Answer:

1) 356
2) 256/5 or 51.2
3) 12543
4) 1/12
5) 14
6) (4-sqrt(2))/3
7) 13
8) -2019000