Gauss School and Gauss Math Circle

2020 Gauss Math Tournament

Division III (Sprint Round 50 minutes, 40 Questions)

Sprint Round

1. Let $f(x) = 2 - x^2 - y^2$. What is $f(5,2)$?

2. If $\frac{a}{b} = \frac{2}{5}$ and $\frac{a+1}{b+1} = \frac{3}{7}$ what is $\frac{a+2}{b+2}$?

3. If x and y are real numbers satisfying $x^2y = 12$ and $xy^2 = 18$, find $x + y$.

4. Let x be a real number such that $x^2 = x+1$. Then $x^3 = ax+b$ for some integers a and b. Find $10a+b$.

5. If a right triangle has area 30 and one of its legs has length 5, what is the length of its hypotenuse?

6. What is the probability that a randomly chosen multiple of 4 between 1 and 100 (inclusive) is a perfect square?

7. What is the maximum number of sections a circle can be divided into with 3 straight lines?

8. A fair coin is flipped four times. What is the probability that at least half of the flips land heads?

9. Let ABC be an acute triangle so that the three angles have integer measures (in degrees). If $\angle A = 45^\circ$ how many values are possible for $\angle B$?

10. How many integers from 1 to 200 (inclusive) can be written in the form $n + \sqrt{n}$ for some positive integer n?

11. Let D be a point on side AC of triangle ABC such that triangles ABC and BDC are similar. If $BC = 6$ and $CD = 4$, find AD.

12. Find the real solution to $20002x^3 + 10001x + 100010 = 20202020$.

13. Compute

$$1 + \frac{1}{5} - \frac{1}{5^2} - \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} - \frac{1}{5^6} - \frac{1}{5^7} + \cdots$$

14. Find the sum of the first 20 integers not divisible by 3.

15. Let $ABCDEF$ be a regular hexagon with side length 6 and ω be a circle centered at A with radius 6. The area of the region inside $ABCDEF$ but outside ω can be written in the form $a\sqrt{3} - b\pi$, where a and b are positive integers. Find $a + b$.

16. We place N *cars* on a 10×10 grid. Each car occupies a single cell and points up, down, left, or right. In a move, we may choose a car and shift it one cell forward to a vacant cell or remove it from the grid if it has reached the edge. Suppose that it is not possible to remove all of the cars from the grid. Find the smallest possible value of N.

17. A box contains two red and two blue marbles. We draw marbles from the box, randomly and without replacement, until we obtain a blue marble. What is the expected number of marbles that we draw?

18. Which prime number can be written in the form $n^2 + 2n - 63$ for some positive integer n?

19. If p, q, r are prime numbers such that $p - q = q - r$ is not divisible by 3 and $q = 37$, what is p?

20. A real number $0 < x < 1$ satisfies

$$2x = 1 - x^2 + x^3 - x^4 + x^5 - x^6 + \cdots .$$

Then x can be written in the form $\frac{\sqrt{a}-b}{c}$, where a, b, c are positive integers and $\gcd(b, c) = 1$. Find $100a + 10b + c$.

21. Let k be an integer for which the quadratic $x^2 + kx + 2020$ has two distinct integer zeroes. Find the number of possible values for k.

22. Find the smallest positive integer n such that

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right)$$

is a perfect cube.

23. Let ABC be a triangle with $AB = AC$ and let D be the midpoint of $\overline{AB}$. If triangle ABC is similar to triangle CDB and $CD = 1$, find AB^2.

24. We place three kings on a 3×3 grid such that no two kings occupy the same square. What is the probability that no two kings attack each other? (Two kings attack each other if the squares they occupy share at least one vertex).

25. If n is a positive integer such that n^5 has 576 divisors, what is the fewest number of divisors that n can have?

26. For how many integers $1 \le n \le 100$ is $n^3 - 1$ divisible by 7?

27. Leo is taking a physics competition. On a certain question, Leo guesses a random real number between 1 and 2 (chosen uniformly). Assume that the correct answer is a random real number between 1 and 3 (chosen uniformly). His answer will be marked correct as long as it is within 10% of the actual answer. What is the probability that he gets the question correct?

28. A line cuts a square into two regions such that the ratio of the areas of the two regions is 1 : 4. The part of the line inside the square is a segment of length L. What is the smallest possible value of L^2?

29. During quarantine, Jeffrey has been playing Spotify almost 24/7, even while he sleeps, so that he can get first place on the "most songs listened to" leaderboard. Jeffrey has a playlist of 181 songs with lengths 2 minutes, 2 minutes 1 second, ... all the way to 5 minutes. Given that after one song is played, the next song is randomly chosen, and Jeffrey starts playing his playlist when he goes to sleep, when he wakes up a long time later, what is the probability that a song 4 minutes or longer is playing?

30. Let x be the largest real number less than 10 satisfying $x + \lfloor x \rfloor^2 = x^2$. Then x can be written in the form $\frac{\sqrt{a}+b}{c}$, where a, b, c are positive integers and $\gcd(b, c) = 1$. Find $a + b + c$.

31. Given that $46^3 - 1$ is divisible by 103, find the integer $46 < n < 103$ such that $n^3 - 1$ is divisible by 103.

32. Let $ABCD$ be a kite with $AB = AD = 3$ and $CB = CD = 7$. A circle ω is inscribed in $ABCD$ (so that ω is tangent to all four sides). Find the largest possible radius of ω.

33. Let S_n denote the sum of the first n positive integers. If $-1 < x < 1$ is a real number satisfying

$$S_1 + \frac{1}{x}S_2 + \frac{1}{x^2}S_3 + \frac{1}{x^3}S_4 + ... = 125,$$

find x.

34. In each cell of a 2×2 grid is an integer with absolute value at most 10. Suppose the grid has the property that for any two numbers in the same row or column, the sum of these two numbers is equal to the product of the other two numbers in the grid. How many possible grids are there?

35. A gambler is playing a game at a casino where 10 coins are flipped, and the gambler wins a dollar if the number of heads is odd, but loses a dollar if the number of heads is even. If the coins are fair, the gambler's expected loss per round is exactly 0. However, the house has rigged the coins! Now, they come up heads with probability $\frac{3}{5}$ and tails with probability $\frac{2}{5}$. Then the gambler's new expected loss per round (in dollars) can be written in the form $a \cdot 10^b$, where a and b are integers and a is not a multiple of 10. Find a.

36. Let ABC be an equilateral triangle. Let D be on $\overline{BC}$ such that $BD = 3$ and $DC = 5$, and let E and F be on $\overline{AC}$ and $\overline{AB}$, respectively, such that triangles BDF and CDE are equilateral. Let N be on the same side of $\overline{EF}$ as A such that triangle NEF is equilateral. Compute NA^2.

37. Find the number of pairs of integers $0 \leq a, b \leq 102$ for which $a^2 + b^2 - 1$ is divisible by 103.

38. Find the largest possible value of

$$ab + ac + ad + bc$$

where a, b, c, d are nonnegative real numbers with $a + b + c + d = 4$.

39. Find the number of ways we can color the squares of a 4×4 grid black and white such that every row and column has an odd number of black squares.

40. Let ABC be a triangle with $AB = 7, BC = 8, CA = 5$ and suppose that O is the circumcenter of triangle ABC. A circle passing through O and A is tangent to $\overline{AB}$. If R is the radius of this circle, find R^2.