# 2022 Gauss Math Tournament Sprint Round (Div. 3)

June 11, 2022

**1**. Let $f(x, y) = x + \frac{y}{x}$. Evaluate $f(1, 2) + f(3, 4)$.

**2**. Andy has 81 peaches, arranged in a $9 \times 9$ square grid. At the center of each unit grid square, he places an additional peach. How many peaches are there in total?

**3**. A survey conducted at Gauss academy concluded that 75% of the students play a sport, 45% play a musical instrument, and 40% of the students that play sports also practice a musical instrument. What percent of the students who do not play sports play a musical instrument?

**4**. There are 121 miles between exit 6 and 26 of the newly built Interstate. If two consecutive exits must be at least 5 miles apart, what is the maximum possible distance between exit 13 and 14?

**5**. Willy creates chairs and spiders. Chairs have no head and four legs. Spiders have one head and eight legs. If Willy has created a total of 2022 heads and 202200 legs, how many chairs did he create?

**6**. Sana has fifty-nine tomatoes. She can trade two tomatoes for five potatoes and six apples, but she cannot trade one tomato at a time. She can then sell each potato for 14 cents and each apple for 5 cents. In dollars, how much can Sana earn?

**7**. Haya wants to find the sum of all interior angles in a convex polygon. He measures every angle and adds them up. However, in this process he mistakenly skips an angle, resulting in an incorrect sum of $2022^\circ$. Find the measure of the missing angle.

**8**. Sana and Haya go birdwatching. Of the 208 birds that flew by, Sana counted 127 of them, while Haya counted 92 of them. Given that 46 birds were counted by neither of the two, how many birds were counted by both?

**9**. Suppose $[a, b]$ represents the average of $a$ and $b$, and $\{a, b, c\}$ represents the average of $a$, $b$, and $c$. What is $\{[0, 1], \{1, 0, 0\}, 1\}$?

**10**. Evaluate $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{36 \cdot 37}$.

**11**. If $2i + 1$ is a root of $x^3 + bx - c$ and $a$ is a root of $x^3 + bx - c$, find $a + 3b + c$ given that $a$, $b$, $c$ are all integers.

**12**. In triangle $\triangle GMT$, $\angle GMT = 150^\circ$, $GM + MT = 26$, and $GM \cdot MT = 159$. Find the area of the triangle.

**13**. Jeff has 32,400 pairs of sunglasses. He wants to distribute them evenly among $X$ people, where $X$ is a positive integer between 10 and 180, inclusive. For how many $X$ is this possible?

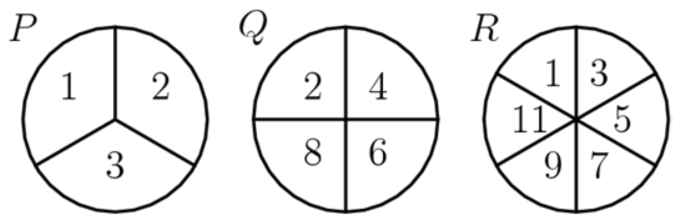
**14**. Let $p$ be the answer to this problem. What is the probability that a number chosen uniformly at random from the interval $[1/2, 10/9]$ is greater than $p$?

**15**. Jimmy and Joey each picked a positive integer. Jimmy's number was 75, while the least common multiple of Jimmy's number and Joey's number was 225. What is the sum of the smallest and largest possible values for Joey's number?

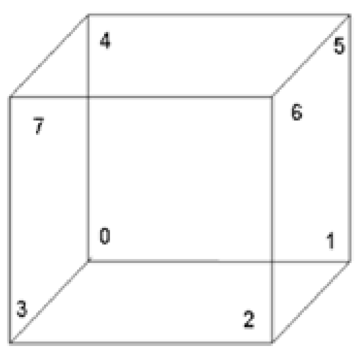
**16**. Chris has a balance scale, as well as five weights weighing 1, 3, 9, 27, and 81 pounds that he can put on either side of the scale. Compute the smallest possible weight, in pounds, that Chris cannot measure.

**17**. Find the sum of all distinct solutions to either $||x|-4|+|2+|x||=x+5$ or $||x|+4|+|2-|x||=x+5$.

**18**. Jeff rotates spinners P, Q, and R and adds the resulting numbers. What is the probability that his sum is an even number?




**19**. A solid unit cube has 8 vertices labeled 1, 2, 3, 4, 5, 6, 7, and 0, as shown below. An ant sits on the surface of the cube and wishes to go from Vertex 1 to Vertex 7. It can only travel on the surface of the cube, but it can freely travel on any face or edge. What is the square of the minimum length of the path the ant travels?



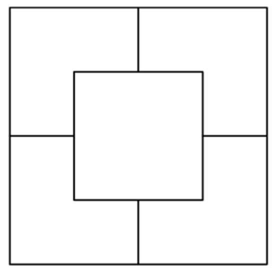

**20**. Polygon $\mathfrak{M}$ has $m$ sides; polygon $\mathfrak{N}$ has $n$ sides. The sum of the interior angles of $\mathfrak{N}$ is $k$ times the sum of the interior angles of $m$, and $n=m+k$. What is the value of $m$, such that $\mathfrak{M}$ and $\mathfrak{N}$ are valid polygons $(m,n\geq 3)$?

**21**. Jerry is taking a true/false test with 25 questions. The test is graded as such: each correct answer yields 4 points, and each incorrect answer takes away one point from the score. The score is out of 100. Leaving questions blank is not allowed. Jerry was lazy and did not study, so he only has a 60% chance of getting any question right on the test. What is his expected score?

**22**. Find the sum of all real solutions to $(x^2-7x+7)^{(x^2+x-2)}=1$.

**23**. An ice cream cone at Chick-Fil-A consists of a truncated cone with an open radius of 40 cm and a closed radius of 20 cm. The height of this cone is $10\sqrt{3}$ cm. Given that ice cream balls at Chick-Fil-A do not deform and the maximum radius of a ball that would be able to make contact with points not on the rim of the cone can be written as $a/\sqrt{b}$. What is $a+b$?

**24**. The area of each of the four congruent L-shaped regions of this 64-inch by 64-inch square is 3/16 of the total area. How many inches long is the side of the center square?



**25**. Anna and Belinda take turns playing archery, and whoever shoots the target first wins. Suppose on each turn, Anna has a 30% chance of shooting the target and Belinda has 60% chance of shooting the target. What is the probability of Anna winning if she shoots first? Express the answer as a common fraction in lowest terms.

**26**. Mason rolls a fair dice 5 times, what is the probability that the sum is exactly 10? Express the answer as a common fraction in lowest terms.

**27**. Given $x$ is an even positive integer what is the greatest positive integer power of 2 that $9^x + 8x + 127$ is always divisible by?

**28**. Miya is on an elevator situated at floor 1 of a five-story building. Every second, if the elevator is on floor $X \leq 4$, it moves down one floor with probability $\frac{X-1}{X+1}$ and moves up one floor with probability $\frac{2}{X+1}$. Once the elevator reaches floor 5, it stays there. What is the probability that Miya reaches floor 5 in at most 6 seconds?

**29**. Three friends, Abby, Bob, and Cassy, each have a number taped to their head so that nobody can see their own number. The three of them make the following remarks:
Abby: "Cassy's number is a multiple of Bob's!"
Bob: "Your numbers add to 120."
Cassy: "Both of your numbers are prime."
Abby: "Aha! I know what my number is!"
Given their numbers are all under 80 and all three friends are perfectly intelligent and truthful, what is Bob's number?

**30**. Primes $p$, $q$, $r$ satisfy $pqr + 2p + 3q + 4r = 123$. Compute $p + q + r$.

**31**. Gauss had no problem finding $S_n = 1 + 2 + 3 + \cdot + n$, but he's having a little more trouble finding $A_n = S_1 + S_2 + S_3 + \cdots + S_n$. And he's having even more trouble finding the sum of $A_n$'s. Help Gauss find the remainder when $A_1 + A_2 + \cdots + A_{100}$ is divided by 1000.

**32**. Let S be the set of all positive integers x for which $2^x + 3^x$ is divisible by 5. Find the remainder when the 2022th smallest element of S is divided by 1000.

**33**. Suppose $f(x) = x - 2$ and $g(x) = 2x^2 + 1$. Consider the set X consisting of all x for which $g(f(x)) = x$. Let the sum of values in X be $a$, and the product of values in X be $b$. What is $a + b$?

**34**. In tetrahedron $ABCD$, $ABC$ is an equilateral triangle with side length $\sqrt{15}$. In addition, $AD = BD = 5$. If the dihedral angle between faces $ABC$ and $ABD$ is 45°, find the surface area of the sphere containing points $A$, $B$, $C$, and $D$. If your answer is $k\pi$, input k.

**35**. What is the largest positive integer that divides $(n+1)^1(n+2)^2(n+3)^3(n+4)^4$ for any (not necessarily positive) integer $n$?

**36**. Brett lives on the bottom left square of Eddy's $6 \times 6$ grid of violins. He wants to reach Eddy, who is at the top right corner. Brett can only jump upward or rightward, while Eddy can only jump leftward and downward. Every second, each of them randomly jumps to a neighboring grid square. Brett and Eddy start jumping simultaneously. If the probability that they meet is $a/b$ in simplest form, what is $a + b$?

**37**. William found that the decimal expansion of $\frac{1}{243}$ has period 27. He writes out the expansion as

$$\frac{1}{243} = 0.\overline{004115 \cdots XYZ},$$

where $XYZ$ is the number formed by the last three digits in the period. Find $XYZ$.

**38**. The operation $\bigoplus$ is defined as follows:

$$A \bigoplus B = \frac{(1 - AB)(A - B)}{(A^2 + 1)(B^2 + 1)}.$$

Find $(1 \bigoplus 2) + (2 \bigoplus 3) + \cdots + (8 \bigoplus 9)$.

**39**. In a civilization far, far away, all real numbers are either *angelic* or *demonic*, adhering to the following rules:

1. $x$ is *angelic* if and only if $x + (1 + 2\sqrt{2})$ is *angelic*.
2. $x$ is *demonic* if and only if $x + (2 + \sqrt{2})$ is *demonic*.

Given that 0 is *angelic*, $3 + \sqrt{2}$ is *demonic*, and $-7\sqrt{2}$ is *demonic*, find the number of *demonic* numbers of the form $A + B\sqrt{2}$, where $A$ and $B$ are (not necessarily positive) integers satisfying $A^2 + B^2 \leq 31$.

**40**. Acute triangle $\triangle ABC$ has $\angle ABC < 45°$. Let $D$ be the point on segment $AC$ such that $BD = DC$, and let $P$ be a point on segment $BD$ such that $AB = PC$. If $\angle APB = 94°$ and $\angle APC = 137°$, find the measure of $\angle BAC$.